AN A PRIORI ESTIMATE FOR THE SINGLY PERIODIC SOLUTIONS OF A SEMILINEAR EQUATION

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Abstract. There exists an exponentially decreasing function f such that any singly 2π-periodic positive solution u of $-\Delta u + u - u^p = 0$ in $[0, 2\pi] \times \mathbb{R}^{N-1}$ verifies $u(x_1, x') \leq f(\|x'\|)$. We prove that with the same period and with the same function f, any singly periodic positive solution of $-\varepsilon^2 \Delta u - u + u^p = 0$ in $[0, 2\pi] \times \mathbb{R}^{N-1}$ verifies $u(x_1, x') \leq f(\|x'\|/\varepsilon)$. We have a similar estimate for the gradient.

1. Introduction.

Let N be an integer, $N \geq 2$, let ε and p be positive real numbers, $p > 1$. We study the equation

$$(1.1) \qquad -\varepsilon^2 \Delta u + u - u^p = 0 \text{ in } S^1 \times \mathbb{R}^{N-1}$$

where $S^1 = [0, 2\pi]$. We mean that u is 2π-periodic in x_1. We consider the positive solutions of (1.1), $u(x_1, x')$ ($x_1 \in S^1$ and $x' \in \mathbb{R}^{N-1}$) that tend to 0 as $\|x'\|$ tends to ∞, uniformly in x_1. It is known that these solutions are radial in x' and decreasing in $\|x'\|$. This can be proved by an application of the moving plane method ([3], [7], [8]). The ground-state solution w_0, defined and radial on $\mathbb{R}^{N-1}$ is a particular solution which does not depend on x_1. In [2], Dancer proved the existence of positive solutions really depending on x_1 and x'. In [1], we studied the case $N = 2$ and we proved the following result:

Theorem 1.1. *(i) The first continuum Σ_1 of positive bounded solutions even in x_1 and x' of (1.1) bifurcating from $(\varepsilon_\star, w_0(x'/\varepsilon_\star))$ is composed of $(\varepsilon_\star, w_0(x'/\varepsilon_\star))$ and of all the solutions (ε, z) of (1.1) such that $z > 0$, z even in x_1 and x_2, $\lim_{x_2 \to \infty} z = 0$ and $\frac{\partial z}{\partial x_1} < 0$ in $]0, \pi[\times \mathbb{R}+$.*
(ii)There exists a bounded subset $\mathcal{A}$ of $L^\infty(S^1 \times \mathbb{R}+)$ such that the set Σ_1 is entirely contained in $]0, \varepsilon_\star] \times \mathcal{A}$.
(iii) For each $(\varepsilon, z) \in \Sigma_1$, z is an isolated point of $\{v \in L^\infty(S^1 \times \mathbb{R}+)$; v even in x_1 and x'; (ε, v) solution of (1.1)$\}$. For every $\varepsilon > 0$, $\varepsilon < \varepsilon_\star$, there exists a finite number of solutions (ε, z) in Σ_1.
(iv) There exists ε_0 such that for all $0 < \varepsilon < \varepsilon_0$ this continuum is a curve that has a one to one $\mathcal{C}^1$ parameterization $\varepsilon \to (\varepsilon, z_\varepsilon)$.

In this paper we suppose that

$$(1.2) \qquad 1 < p < \frac{N+2}{N-2}$$

If $N = 2$, this condition is $p > 1$.

We know, see [2], that the condition (1.2) for p is a necessary and sufficient condition to have the following property : There exists $M > 0$ such for all $\varepsilon > 0$, any positive solution u of (1.1) verifies

$$\|u\|_{L^\infty} \leq M \tag{1.3}$$

This property is related to the nonexistence of positive solutions for the equation $-\Delta u - u^p = 0$, more precisely

$$(v \geq 0 \quad , \quad -\Delta v - v^p = 0 \text{ in } \mathbb{R}^N) \Rightarrow (v = 0) \tag{1.4}$$

(see Gidas and Spruck [4]). This paper is devoluted to some a priori estimates for the solutions of (1.1).

Theorem 1.2. *There exists a real number K independent of $\varepsilon > 0$ and of any solution u of (1.1), such that for all $x = (x_1, x')$ in $S^1 \times \mathbb{R}^{N-1}$, we have, with $r' = \|x'\|$,*

$$u(x) \leq K e^{-\frac{r'}{\varepsilon}} \left(\frac{r'}{\varepsilon}\right)^{\frac{2-N}{2}} \tag{1.5}$$

$$\|\nabla u(x)\| \leq \frac{K}{\varepsilon} e^{-\frac{r'}{\varepsilon}} \left(\frac{r'}{\varepsilon}\right)^{\frac{2-N}{2}} \tag{1.6}$$

In [1], we have proved (1.5) for $N = 2$ but with a constant K depending on the solution (ε, u) for ε greater than some $\overline{\varepsilon} > 0$. Our proof extends easily for $N \geq 2$ and for the derivatives of u. We have now to prove that K is independent from the solution (ε, u), even when ε tends to 0.

In all what follows we will use $\tilde{u}(x_1, x') = u(\varepsilon x_1, \varepsilon x')$ for $(x_1, x') \in S^1/\varepsilon \times \mathbb{R}^{N-1}$. The notation Δ' will stand for the Laplacian operator in $\mathbb{R}^{N-1}$.

2. Proof of Theorem 1.2

We begin the proof by two propositions.

Proposition 2.1. *Let v be a bounded solution of*

$$-\Delta v + v - v^p = 0 \text{ in } \mathbb{R}^2 \tag{2.7}$$

Let us suppose that $\frac{\partial v}{\partial x_i}$ is bounded and $\frac{\partial v}{\partial x_i} \leq 0$ in $\mathbb{R}^2$, for $i = 1$ or $i = 2$. Then v does not depend on the variable x_i.

Proof: From Ghoussoub-Gui, [6], Theorem 1.1, there exist a function U and a vector $a \in \mathbb{R}^2$ such that $v(x) = U(a.x)$. We have

$$-\|a\|^2 U''(a.x) + (U - U^p)(a.x) = 0$$

If $a_i \neq 0$, U is monotone and bounded in $\mathbb{R}$. The only possibility is that U is a constant function, equal to 0 or 1, so is v.

Proposition 2.2. *Let (ε, u) be solutions of (1.1). Then $\tilde{u}(x_1, r)$ tends to 0 as r tends to ∞, uniformly with respect to x_1 and to ε and u.*

Proof In this proof, we omit the indices of the sequences. Let us suppose, by contradiction, that there exist a sequence $(a, b) \in \mathbb{R}^N$, with $\|b\|$ tending to $+\infty$, a real positive number ε_2 and solutions (ε, u) of (1.1) such that $\tilde{u}(a, b) \geq \varepsilon_2$. We can suppose $\varepsilon_2 < 1$. For every solution (ε, u), we have that $\lim_{r \to \infty} \tilde{u}(x_1, r) = 0$, uniformly in x_1. So, for every $\varepsilon_1 \in]0, \varepsilon_2[$, there exists a sequence, $\overline{b}$, $\|\overline{b}\| \geq \|b\|$, such that $\tilde{u}(a, \overline{b}) = \varepsilon_1$. With the same argument, we define a sequence, still denoted by b, with $\|b\|$ tending to ∞, such that $\tilde{u}(a, b) = \varepsilon_2$. As $\tilde{u}$ is radial in x', let us define

$$v(x_1, r) = \tilde{u}(x_1 + a, r + \|b\|) \text{ for } r \geq -\|b\|$$

and

$$\overline{v}(x_1, r) = \tilde{u}(x_1 + a, r + \|\overline{b}\|) \text{ for } r \geq -\|\overline{b}\|$$

The function v verifies

$$-v_{x_1 x_1} - v_{rr} - \frac{N-2}{r + \|b\|} v_r + v - v^p = 0$$

and $\overline{v}$ verifies a similar equation. It is standard that the both sequences v and $\overline{v}$ tend uniformly on the compact sets of $\mathbb{R}^2$ to limits, which will be denoted respectively by z and $\overline{z}$. But z and $\overline{z}$ are positive, bounded and non increasing in the variable r and they are periodic in x_1. Moreover, z and $\overline{z}$ verify

$$-z_{x_1 x_1} - z_{rr} + z - z^p = 0 \text{ in } \mathbb{R}^2$$

By Proposition 2.1, z and $\overline{z}$ depend only on x_1. By Kwong, [9], if they are not constant functions, they oscillate indefinitely as x_1 tends to ∞, around the solution 1. As $0 < \varepsilon_1 < \varepsilon_2 < 1$, then $\overline{z}$ and z are not constant solutions. So z and $\overline{z}$ oscillate infinitely around 1, too. The function $h = z - 1$ and the function $\overline{h} = \overline{z} - 1$ verify respectively the equations

$$h'' + h\left(-1 + \frac{z^p - 1}{z - 1}\right) = 0 \quad \text{and} \quad \overline{h}'' + \overline{h}\left(-1 + \frac{\overline{z}^p - 1}{\overline{z} - 1}\right) = 0 \tag{2.8}$$

As $z \geq \overline{z}$ and $z(0) > \overline{z}(0)$, we have from the ordinary differential equations theory that $z > \overline{z}$. It is easy to see that $-1 + \frac{z^p - 1}{z - 1} > -1 + \frac{\overline{z}^p - 1}{\overline{z} - 1}$. By the Sturm Theory (see Ince, quoted in [9] , Lemma 1), applied to the equations (2.8), there exists at least a zero of $z - 1$ between any two consecutive zeroes of $\overline{z} - 1$. But there exist pairs (α, β) of zeroes of $\overline{z} - 1$ such that $\overline{z} > 1$ in $]\alpha, \beta[$. Thus $z > 1$ in $[\alpha, \beta]$. We get a contradiction. We infer that the sequence (a, b), in the beginning of this proof, doesn't exist. We have proved the proposition.

We will need the following lemma

Lemma 2.1. *There exists M, such that for all solution (ε, u) of (1.1)*

$$\|\nabla \tilde{u}\|_{L^\infty(\frac{S^1}{\varepsilon} \times \mathbb{R}^{N-1})} \leq M \tag{2.9}$$

Proof Let $(a, b) \in (S^1/\varepsilon) \times \mathbb{R}^{N-1}$. We set $v(x_1, x') = \tilde{u}(x_1 + a, x' + b)$. It verifies $-\Delta v + v - v^p = 0$ in $(S^1/\varepsilon) \times \mathbb{R}^{N-1}$. Moreover, we have $\|v\|_\infty \leq M$, for a constant M independent from ε. By standart elliptic arguments, [5], ∇v is bounded on the compact sets of $\mathbb{R}^N$. So, there exists M, independent from ε, such that $\|\nabla v(0, 0)\| \leq M$. This proves (2.9).

Proof of Theorem 1.2 We define

$$h(r') = \int_0^{2\pi/\varepsilon} \tilde{u}(x_1, r')dx_1$$

There exists a constant C, independent from the solution (ε, u), such that $\|h\|_{L^\infty(\mathbb{R}^{N-1})} \leq \frac{C}{\varepsilon}$. Since $u \to 0$, uniformly in x_1, as r' tends to ∞, then for all $\eta < 1$, there exists $X > 0$ such that for all $r' > X$ and for all ε we have for all solution (ε, u) and for all $x_1 \in \frac{S^1}{\varepsilon}$

$$\tilde{u}^{p-1}(x_1, r') < \eta \tag{2.10}$$

Integrating (1.1) with respect to x_1, we find for $r' > X$

$$h_{rr} + ((N-2)/r')h_r > (1-\eta)h \tag{2.11}$$

Let us multiply (2.11) by h_r, we obtain that the function $h_r^2 - (1-\eta)h^2$ is non increasing. Moreover, it tends to 0 as r' tends to ∞. We get $h_r + \sqrt{1-\eta}h \leq 0$, for $r' > X$. So there exists C such that for all r'

$$h(r') \leq \frac{C}{\varepsilon} e^{-\sqrt{1-\eta}r'} \tag{2.12}$$

Let us remark that the constant C is independent from the choice of the solution (ε, u).

Let $R > 0$ be a given positive real number. We use a Harnack inequality ([5], Theorem 9.20) to get that there exists a constant C independent from y and from ε such that

$$\sup_{B_R(y)} \tilde{u} \leq C \int_{B_{2R}(y)} \tilde{u} \leq C \int_{\|x'-y'\| \leq 2R} \int_{y_1-R}^{y_1+R} \tilde{u}(x_1, x')dx_1 dx' \tag{2.13}$$

that gives

$$\sup_{B_R(y)} \tilde{u} \leq C \int_{\|x'-y'\| \leq 2R} h(\|x'\|)dx'.$$

Finally, using (2.12), for all $\eta \in]0, 1[$ there exists C, independent from the solution (ε, u), such that,

$$\tilde{u}(y) \leq \frac{C}{\varepsilon} e^{-\eta\|y'\|} \tag{2.14}$$

For the remainder of the proof, we will need the Green function for the equation (1.1). We have

$$G(x_1,x') = \sum_{j=0}^{\infty} \frac{k_j^{N-3}}{\varepsilon^{N-1}} g(\frac{k_j}{\varepsilon}x')\cos(jx_1) \tag{2.15}$$

where $k_j = \sqrt{1+\varepsilon^2 j^2}$ and g is the Green function for the operator $-\Delta' + I$ in $\mathbb{R}^n$, $n = N-1$, with the null limit at infinity. It is recalled in [3] that

$$0 < g(r) \le C\frac{e^{-r}}{r^{n-2}}(1+r)^{(n-3)/2} \text{ for } n \ge 2 \text{ and } g(r) = \frac{1}{2}e^{-r} \text{ for } n = 1 \tag{2.16}$$

We will need the following estimate, valid for all $\eta \in]0,1[$.

$$\int_{\mathbb{R}^{N-1}} g(\|y'-x'\|)e^{-\eta\|y'\|}dy' \le Ce^{-\eta\|x'\|} \tag{2.17}$$

which is an easy consequence of (2.16). For all function f, that is 2π-periodic in x_1, the solution of

$$-\varepsilon^2\Delta u + u = f \text{ in } \mathbb{R}^N$$

that is 2π-periodic in x_1 and that tends to 0, as $\|x'\|$ tends to ∞ is $u = G\star f$. If f is positive, then u is positive, by the maximum principle. So G is positive. Moreover we can use (2.15) to verify that

$$\int_{S^1} G(x_1,x')dx_1 = \frac{2\pi}{\varepsilon^{N-1}} g(\frac{x'}{\varepsilon}) \tag{2.18}$$

Let us prove that for all $\eta \in]0,1[$, there exists C, independent from x_1 and from (ε, u) such that

$$\tilde{u}(x_1,x') \le Ce^{-\eta r'} \tag{2.19}$$

It is clear by (2.14) that for all solution (ε, u) and all $\eta \in]0,1[$, the function $\tilde{u}e^{\eta r}$ belongs to $L^\infty(\mathbb{R}^N)$. We set

$$K(\eta) = \|\tilde{u}e^{\eta r'}\|_\infty$$

We use the Green function G to get

$$u(x_1,x') = \int_{S^1\times\mathbb{R}^{N-1}} G(y_1-x_1, y'-x')u^p(y_1,y')dy_1dy' \tag{2.20}$$

and (2.18) gives

$$\tilde{u}(x_1,x') \le 2\pi K\Big(\frac{\eta}{p}\Big)^p \int_{\mathbb{R}^{N-1}} g(\|y'-x'\|)e^{-\eta\|y'\|}dy'$$

By (2.17), we infer that there exists a constant C, independent from (ε, u), such that

$$K(\eta) \le CK\Big(\frac{\eta}{p}\Big)^p \tag{2.21}$$

Now, let $\tau = (\tau_1, \tau')$ be such that the function $\tilde{u}(x+\tau)e^{\eta\|x'+\tau'\|}$ attains its maximal value at $x = 0$. The existence of τ is provided by (2.14). Let us suppose that $K(\eta)$

tends to ∞. We claim that $\|\tau'\|$ tends to infinity. Let us prove this claim. Let α be a positive real number, that will be chosen later. We set

$$v(x) = \tilde{u}(\alpha x + \tau)e^{\eta\|\alpha x' + \tau'\|}/K(\eta)$$

It verifies

$$-\Delta v + \Big(1 + \eta^2 + \frac{(N-2)\eta}{\|\alpha x' + \tau'\|}\Big)\alpha^2 v$$

$$= K(\eta)^{p-1}\alpha^2 e^{(-p+1)\eta\|\alpha x'+\tau'\|}v^p + \frac{2\eta\alpha^2}{K(\eta)}\sum_{i=2}^{N}\frac{\partial\tilde{u}}{\partial x_i}(\alpha x + \tau)\frac{(\alpha x_i + \tau_i)}{\|\alpha x' + \tau'\|}e^{\eta\|\alpha x'+\tau'\|}$$

If $\|\tau'\|$ were bounded, we would choose α that tends to 0 such that $K(\eta)^{p-1}\alpha^2 e^{(-p+1)\eta\|\alpha x'+\tau'\|}$ tends to 1. By Lemma 2.1 and by standard results, v would tend to a limit $\overline{v}$, uniformly in the compact sets of $\mathbb{R}^N$. Then, $\overline{v}$ would verify $-\Delta\overline{v} - \overline{v}^p = 0$ while $0 \le \overline{v} \le 1$ and $\overline{v}(0) = 1$. This is impossible by (1.4). So, if we suppose that $K(\eta)$ tends to ∞, then $\|\tau'\|$ tends to ∞. Let $\tilde{\tau} = (\tilde{\tau}_1, \tilde{\tau}')$ be such that $K(\frac{\eta}{p}) = \tilde{u}(\tilde{\tau})e^{\frac{\eta}{p}\|\tilde{\tau}'\|}$. We have $K(\frac{\eta}{p})^p = \tilde{u}^p(\tilde{\tau})e^{\eta\|\tilde{\tau}'\|}$, that gives

$$K\Big(\frac{\eta}{p}\Big)^p \le K(\eta)\tilde{u}^{p-1}(\tilde{\tau}) \tag{2.22}$$

Then (2.21) and (2.22) give

$$K(\eta) \le CK\Big(\frac{\eta}{p}\Big)^p \le CK(\eta)\tilde{u}^{p-1}(\tilde{\tau}) \tag{2.23}$$

Consequently, if $K(\eta)$ tends to ∞, then $K(\frac{\eta}{p})$ tends to ∞, too. Then, $\|\tilde{\tau}'\| \to \infty$. By Proposition 2.2, we have $\tilde{u}(\tilde{\tau}) \to 0$. Then (2.23) gives a contradiction. So, we have proved that for all $\eta \in]0,1[$, $K(\eta)$ is bounded, independently from (ε, u). We have (2.19). Now, let us choose η such that $\eta p > 1$. In [3], it is proved that for $b > 1$ and for $N - 1 \ge 2$

$$\int_{\mathbb{R}^{N-1}} g(\|x' - y'\|)e^{-b\|y'\|}dy' \le C\|x'\|^{\frac{2-N}{2}}e^{-\|x'\|} \tag{2.24}$$

We can use (2.16) to prove that the estimate (2.24) is valid also for $N = 2$. Now we use (2.20), (2.18) and (2.24) to obtain (1.5) with K independent from (ε, u). Now, let us estimate the gradient of u. We have, for $i = 1, ..., N$

$$\frac{\partial u}{\partial x_i}(x_1, x') = p\int_{S^1\times\mathbb{R}^{N-1}} G(y_1 - x_1, y' - x')(u^{p-1}\frac{\partial u}{\partial x_i})(y_1, y')dy_1dy' \tag{2.25}$$

Since $\frac{\partial\tilde{u}}{\partial x_i}$ is bounded and $u \le Ce^{-r'/\varepsilon}$, that gives

$$\mid \frac{\partial u}{\partial x_i}(x_1, x') \mid \le \frac{C}{\varepsilon}\int_{S^1\times\mathbb{R}^{N-1}} G(y_1 - x_1, y' - x')e^{-(p-1)\|y'\|/\varepsilon}dy_1dy'$$

and (2.18) gives

$$\mid \frac{\partial\tilde{u}}{\partial x_i}(x_1, x') \mid \le C\int_{\mathbb{R}^{N-1}} g(\|y' - x'\|)e^{-(p-1)\|y'\|}dy' \tag{2.26}$$

Now, the proof is more easy if $p > 2$ than if $p < 2$. If $p > 2$, we deduce directly (1.6) from (2.24) and (2.26). If $1 < p < 2$, we deduce from (2.17) and (2.26) that

$$| \frac{\partial \tilde{u}}{\partial x_i}(x_1, x') | \leq Ce^{-(p-1)\|x'\|}$$

Iterating this process, we get an integer k such that

$$| \frac{\partial \tilde{u}}{\partial x_i}(x_1, x') | \leq Ce^{-k(p-1)\|x'\|}$$

with $k(p-1) < 1$ and $(k+1)(p-1) \geq 1$. If $(k+1)(p-1) > 1$, we get (1.6) and the proof is complete. If $(k+1)(p-1) = 1$, we get

$$| \frac{\partial \tilde{u}}{\partial x_i}(x_1, x') | \leq C \int_{\mathbb{R}^{N-1}} g(\|y' - x'\|)e^{-\|y'\|}dy' \tag{2.27}$$

If $N \geq 3$, we have

$$\int_{\mathbb{R}^{N-1}} g(\|y'-x'\|)e^{-\|y'\|}dy' \leq C \int_{\mathbb{R}^{N-1}} e^{-\|x'-y'\|-\|y'\|}(1+\|x'-y'\|)^{(N-4)/2}/\|x'-y'\|^{N-3}dy'$$

We can write the integral in the right hand member of this inequality as $I = I_1 + I_2$ and

$$I_1 = \int_{\|z\|\leq\|x'\|} e^{-\|z\|-\|z+x'\|}(1 + \|z\|)^{(N-4)/2}/\|z\|^{N-3}dz$$

and

$$I_2 = \int_{\|z\|\geq\|x'\|} e^{-\|z\|-\|z+x'\|}(1 + \|z\|)^{(N-4)/2}/\|z\|^{N-3}dz$$

We obtain, as $\|x'\|$ tends to ∞,

$$I_1 \leq e^{-\|x'\|} \int_0^{\|x'\|} (1 + s)^{\frac{N-4}{2}} sds \quad \text{ and } \quad I_2 \leq e^{\|x'\|} \int_{\|x'\|}^{+\infty} e^{-2s}(1 + s)^{\frac{N-4}{2}} sds$$

These integrals are both less than $Ce^{-\|x'\|}\|x'\|^{\frac{N}{2}}$. Thus, if $N \geq 3$ we have obtained that

$$| \frac{\partial \tilde{u}}{\partial x_i}(x_1, x') | \leq Ce^{-\|x'\|}\|x'\|^{\frac{N}{2}} \tag{2.28}$$

If $N = 2$, we have, when $|x'|$ tends to ∞

$$\int_{\mathbb{R}} g(|y' - x'|)e^{-|y'|}dy' \leq C \int_{\mathbb{R}} e^{-|x'-y'|-|y'|}dy' \leq C|x'|e^{-|x'|}$$

In any case, we get that there exists $b \in]0, 1[$, with $b + p - 1 > 1$ and such that $| \frac{\partial \tilde{u}}{\partial x_i}(x_1, x') | \leq Ce^{-b\|x'\|}$ Using this estimate in (2.25) and thanks to (2.24), we get (1.6), for $1 < p < 2$. If $p = 2$, (2.26) is (2.27) and we deduce (2.28) again. This ended the proof of Proposition 1.2.

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